

Chaos and Noise

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Simple dynamical systems – with a small number of degrees of freedom – can behave in a complex manner due to the presence of chaos. Such systems are most often (idealized) limiting cases of more realistic situations. Isolating a small number of dynamical degrees of freedom in a realistically coupled system generically yields reduced equations with terms that can have a stochastic interpretation. In situations where both noise and chaos can potentially exist, it is not immediately obvious how Lyapunov exponents, key to characterizing chaos, should be properly defined. In this paper, we show how to do this in a class of well-defined noise-driven dynamical systems, derived from an underlying Hamiltonian model.

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I. INTRODUCTION

The characterization of deterministic chaos, particularly in Hamiltonian systems, is now well-established, and essentially consists of computing and understanding the Lyapunov spectrum of the dynamical system [1, 2]. In the vast majority of experiments, however, Hamiltonian systems are only an idealization, and it is not always clear – given a phenomenological description – what role dynamical chaos might actually play in the physics, let alone be certain about how to characterize it.

A somewhat similar situation also exists in the modeling of complex systems, where often one wishes to separate ‘slow’ and ‘fast’ degrees of freedom in such a way that the fast degrees of freedom can be viewed as a stochastic forcing term, often called ‘noise’ [3–5]. As emphasized by Zwanzig [6], a key problem is that depending on the nature of the approximations one chooses to make, the resulting noise terms in the stochastic dynamical equations (Langevin equations), can end up being very different.

It is therefore easy to appreciate that when both noise and chaos are combined, the situation is doubly complicated. Nevertheless, such situations are not only common, but also of significant interest in many applications. These include such diverse areas as noise-induced chaos [7–10], ecology [11], galactic dynamics [12], and the quantum-classical transition [13, 14].

The purpose of this paper is to consider the interaction of noise and chaos in a dynamical model where the basic physical and approximation-related issues can be separately understood. This will not only help clarify the nature of some previous disagreements in the literature but also provide a more well-founded notion of the (maximal) Lyapunov exponent (LE) in noisy systems, at least within a specific, well-defined, context.

The overall picture adopted here assumes the existence of a complex Hamiltonian system, where one focuses on a small set of relevant degrees of freedom, deriving effective

equations for their evolution. Because the full system is specified, issues of how to define chaos and noise, and their interaction, are easier to clarify. Discussions of the possible difficulties and choices of definitions in the context of a more phenomenological setting can be found in Refs. [15, 16].

In this paper, we consider Hamiltonian models for Brownian motion as the relevant archetype. In these models, a dynamical system is coupled to a heat bath, modeled by an ensemble of non-interacting harmonic oscillators. An analogous model was first described by Rubin [17], and the basic notion was later elucidated by a number of authors, including Ford, Kac, and Mazur [18], Zwanzig [19], and, as better known in the quantum context, by Caldeira and Leggett [20]. Although these models are by no means completely general, they provide an excellent basis for addressing conceptual problems.

In adopting this view we nevertheless wish to be clear in what circumstances analyses such as ours are meant to apply: Our results will not be directly relevant to heuristic models of complex systems that are not based on a first-principles analysis, nor are they intended to apply in the strong noise limit, where the effects of chaos can be washed out by the effects of the noise drive.

As a consequence of the self-consistent nature of our analysis, the (nonequilibrium) fluctuation-dissipation theorem will ensure that system trajectories explore a unique distribution at late times, that defined by thermal equilibrium, $f_{eq} \sim \exp\{-\beta H\}$, where H is the system Hamiltonian and $\beta = 1/k_B T$ is proportional to the inverse temperature. Thus the LE we consider is averaged over the canonical distribution, in contrast to a constant energy hypersurface in the non-noisy case, when the system is not coupled to a heat bath. The lower the temperature of the heat bath, the smaller the noise strength, and the smaller the energy fluctuations. In the singular limit of vanishing coupling to the heat bath, any phase space distribution that is a function of the system

Hamiltonian is invariant, and there is no longer the notion of a single unique distribution that can be used to study trajectories in the late-time limit. However, in this case, as a consequence of the ergodic theorem, constant energy hypersurfaces are fully explored by long-time trajectories, and, as is familiar, the LEs become functions of the energy.

At this point it is important to distinguish between ‘external’ and ‘internal’ noise. The case of external noise arises when the noise drive is unaffected by the system evolution, while internal noise refers to the fluctuations of a complex system coupling to its own slow modes; in this case, the noise may be affected by the system evolution, both in terms of its intrinsic statistical properties and in terms of the particular noise realization associated with a background trajectory (even if the statistics are unaffected). We discuss both of these cases below.

Finally, it is important to consider various limits, such as the overdamped (strong-coupling) limit and the weak-noise (low temperature) limit. Note that these limits are independent, but can define physical timescales relevant to our analysis. In the examples considered here, the system relaxation time is independent of temperature or noise strength and depends only on the damping coefficient, in turn set by the inverse of the system-bath coupling strength. The convergence timescales for the LE are typically very long, significantly longer than typical thermal relaxation timescales, consequently the initial phase of the evolution is unimportant. Thus, for our purposes, the relevant dynamics of the system is that related to the exploration of a thermal equilibrium state as mentioned above. In the cases we study, the equilibrium is established by interaction with a heat bath (equilibrium for a canonical ensemble, as given by the static solution of a Fokker-Planck equation), satisfying the fluctuation-dissipation relation. Energy fluctuations are bounded as long as the system potential is asymptotically confining. For the most part we will consider the low noise case but discuss both the weak and strong coupling limits. The strong coupling limit is singular and care is needed with the analysis in order to avoid incorrect results.

The rest of the paper is organized as follows. We begin in Section II with a short review of the derivation of the relevant Langevin equations, emphasizing issues that will be taken up in later sections. We will then consider the case of the Langevin equation with the noise treated entirely as an external perturbation (Section III), define the LE, and proceed to show – without making the overdamped approximation – that when the system is one-dimensional, the addition of noise cannot lead to a positive LE. This changes the result of Ref. [21] who claimed that in this case, the LE could be positive (noise-induced chaos) or negative. Extending to higher dimensions, we show that in this case, the LE can indeed be positive or negative, in contrast to Ref. [22], who claimed that the LE is always negative. Having completed the analysis for the external noise case, we then extend our analysis to systems with internal noise (Section IV). Examples of

numerical results that agree with our conclusions can be found in Refs. [12–14]. Finally, we discuss some open questions and possible future research directions related to chaos and noise in our conclusion.

For notational consistency we will write all our vectors in boldface and all our matrices in scripts, and the components of vectors and matrices will be in normal font, i.e. v_1 is the first component of the vector \mathbf{v} and M_{11} is the top-left entry of the matrix \mathcal{M} .

II. INDEPENDENT OSCILLATOR MODEL FOR BROWNIAN MOTION

As discussed in the Introduction, for the most part, stochastic differential equations appear in the modeling of physical systems primarily in a phenomenological context. Because of the uncontrolled approximations inherent in how these equations are often written down, it is not obviously apparent how to think about the interaction of noise and chaos in a systematic fashion, as both of these can be very sensitive to choices made in the modeling process, which may or may not be entirely self-consistent, or even physically correct.

There are, nevertheless, several examples of areas where stochastic equations can be derived in a more or less controlled fashion (based on assumptions such as timescale separation), starting from an initial Hamiltonian formulation. These include the (multiplicative noise) Langevin description of the Landau equation in plasma physics [23], the description of Brownian motion based on a Hamiltonian system coupled to a large set of independent oscillators [17–20], and, more generally, equations derived using the Mori-Zwanzig projection operator technique [24–26].

To fix ideas, we present here a short derivation of Brownian motion using the independent oscillator model, following Zwanzig [6]. The full (positive-definite) Hamiltonian, of a system interacting with a “bath” of oscillators is taken to be:

$$H = \frac{1}{2}p^2 + V(x) + \frac{1}{2} \sum_j \left[p_j^2 + \omega_j^2 (q_j - a_j(x))^2 \right], \quad (1)$$

where the system coordinates are (x, p) and the smooth functions $a_j(x)$ describe possibly nonlinear couplings of the system to the oscillators. Note that there is an assumption here that no degree of freedom of the bath is strongly perturbed by the system of interest; this justifies treating the q_j effectively in a linearized approximation. The distribution of the frequencies, ω_j , as well as the choice of the function $a_j(x)$ controls the noise memory as discussed below.

To simplify matters, we first consider a linear coupling to the oscillators, i.e., we set $a_j(x) = (\gamma_j/\omega_j^2)x$. Writing down Hamilton’s equations starting from Eq. (1), formally solving the equations of motion for the oscillators treating $\gamma_j x(t)$ as time-dependent external force, and substituting this solution into the equations for the

system variables, we obtain the formal Langevin equation,

$$\dot{p}(t) = -V'(x) - \int_0^t K_N(t-s)p(s) + F_N(t), \quad (2)$$

where the damping kernel is

$$K_N(t-s) \equiv \sum_j \frac{\gamma_j^2}{\omega_j^2} \cos \omega_j(t-s), \quad (3)$$

and the ‘noise’ force is

$$F_N(t) = \sum_j \gamma_j \left[q_j(0) - \frac{x(0)}{\omega_j^2} \gamma_j \right] \cos \omega_j t + \sum_j \gamma_j p_j(0) \frac{\sin \omega_j t}{\omega_j t}. \quad (4)$$

The damping kernel, or memory function, is in general not Markovian and is determined entirely by the coupling constants and oscillator frequencies. In contrast, $F_N(t)$ is determined entirely by the initial conditions. If these are known precisely then $F_N(t)$ is not a noise term. However, if the initial conditions are prescribed in a statistical manner, then the situation is different. Suppose that the statistical ensemble of initial conditions is such that the first moments vanish, i.e.,

$$\begin{aligned} \langle p_j(0) \rangle_0 &= 0, \\ \langle q_j(0) - \frac{\gamma_j}{\omega_j^2} x(0) \rangle_0 &= 0, \end{aligned} \quad (5)$$

then, it is immediate from Eq. (4) that $\langle F_N(t) \rangle_0 = 0$. Now, if the second moments of the initial ensemble are,

$$\begin{aligned} \langle p_j(0)p_k(0) \rangle_0 &= k_B T \delta_{jk}, \\ \langle [q_j(0) - \frac{\gamma_j}{\omega_j^2} x(0)][q_k(0) - \frac{\gamma_k}{\omega_k^2} x(0)] \rangle_0 &= \frac{k_B T}{\omega_j^2} \delta_{jk}, \end{aligned} \quad (6)$$

then,

$$\langle F_N(t)F_N(t') \rangle_0 = k_B T K_N(t-t'), \quad (7)$$

which is a generalized fluctuation-dissipation theorem. An initial ensemble that satisfies these conditions is one in which the heat bath is in thermal equilibrium with respect to the system, whereas the system variables are allowed to have an arbitrary distribution:

$$f(t=0) \sim f_0^{sys}(x, p) \exp(-\beta H_{bath}), \quad (8)$$

where H_{bath} represents the Hamiltonian of the oscillators and the system-bath couplings, i.e., the term containing the summation in Eq. (1). In general, the noise is not Markovian; however, in the special limiting case of a large number of oscillators, N , following a Debye distribution (i.e., the spectral distribution $g(\omega) = 3\omega^2/\omega_D^2$ for $\omega < \omega_D$, and $g(\omega) = 0$ for $\omega > \omega_D$), and assuming that the

system momentum varies slowly on timescales set by the inverse Debye cutoff $\tau_D = 1/\omega_D$, the kernel $K_N(t-s)$ can be approximated as a delta function, and the Langevin equation takes on the more familiar form

$$\dot{p}(t) = -V'(x) - \lambda p(t) + F_N(t), \quad (9)$$

where the noise is ‘white’ due to the sharpness of the memory kernel, and it is Gaussian thanks to the quadratic nature of H_{bath} in Eq. (8). Here, the damping coefficient is $\lambda = 3\pi\gamma^2/2\omega_D^2$, where we set $\gamma_j \rightarrow \gamma/\sqrt{N}$ when taking the limit of a large number of oscillators. The fluctuation-dissipation relation (7) now becomes

$$\langle F_N(t)F_N(t') \rangle_0 = 2\lambda k_B T \delta(t-t'). \quad (10)$$

It is important to point out here that the simplified derivation given above can be considerably sharpened: The system trajectories can be rigorously shown to converge to the solutions of a stochastic problem both in the weak [27] and strong sense [28].

Returning to the case of an arbitrary nonlinear coupling specified by $a_j(x)$, the above analysis goes through essentially unchanged, but with a more complicated memory kernel associated with multiplicative noise (albeit, still white and Gaussian in the case of the Debye spectrum) that continues to satisfy the fluctuation-dissipation theorem. The presence of multiplicative noise can cause qualitatively new dynamical effects because the noise amplitude depends on the system variables. Such effects include modifications of the equilibration rate [29, 30] and the existence of long-time tails in transport theory [31]. The basic procedure described in this paper can be extended to the case of multiplicative noise as long as the system is being analyzed in the asymptotic late-time limit, i.e., on timescales much longer than the relaxation time.

Although this class of models is relatively simple, it can be extended in interesting directions by changing the system potential or by manipulating the spectral distribution of bath oscillators. Examples of such extensions include studies of escape problems and tests of transition state theory [32, 33] and Hamiltonian models leading to fractional kinetics [34].

In the models discussed above, the fluctuation-dissipation theorem follows as a consequence of how we chose the initial condition for the bath oscillators. This choice corresponds to running many copies of the system drawn from some initial distribution but, in which, for each realization of the initial condition, the oscillator bath is in thermal equilibrium with respect to $x(0)$. The system following the Langevin equation (9) will be driven at late times to the thermal equilibrium distribution $f_{eq} \sim \exp(-\beta H_{sys})$, where H_{sys} corresponds to the first two terms of the full Hamiltonian H , as specified in Eq. (1). (This result can be most easily derived by considering the associated Fokker-Planck equation for the phase space distribution [35].) Consequently, time-averaged quantities exist ($\bar{f}_\tau(x, p) = 1/\tau \int_0^\tau f(x, p; t) dt$),

and are stable in the limit $\tau \rightarrow \infty$. We note that we are explicitly not considering systems with an external time-drive in which case there is, in principle, no equilibrium state.

Note also that the Langevin equation derived here is inherently second-order in time (or, more generally, an even number of first-order equations). In the limit of strong coupling (large λ), the velocity can be eliminated as a ‘fast’ variable and, in leading approximation, a first-order stochastic equation may be derived [36]. This limit is singular, however, and care must be taken in applying it in different situations.

Finally, we turn to a discussion of how to define LEs for the system considered here. In principle, there is no problem, since given the full Hamiltonian and prescribed initial conditions, we can consider a small perturbation of the initial conditions around some fiducial trajectory of the (full) coupled system. The summary of this procedure is as follows. For a $2n$ -dimensional dynamical system governed by a set of evolution equations, $d\mathbf{z}/dt = \mathbf{F}(\mathbf{z}, t)$, where $\mathbf{z} = (z_1, z_2, \dots, z_{2n})^T$ (similarly for \mathbf{F}), consider (i) a fiducial trajectory, $\mathbf{z}_0(t)$, (ii) define deviations from it via $\mathbf{Z} = \mathbf{z} - \mathbf{z}_0$, and (iii) linearize the original set of equations, yielding,

$$\frac{d\mathbf{Z}}{dt} = \mathcal{D}\mathcal{F}(\mathbf{z}_0, t) \cdot \mathbf{Z}, \quad (11)$$

where $\mathcal{D}\mathcal{F}$ is the $2n \times 2n$ Jacobian matrix. The tangent map $\mathcal{Q}(\mathbf{z}_0(t), t)$ is found by integrating the linearized equations along the fiducial trajectory; $\mathcal{Q}(\mathbf{z}_0(t), t)$ evolves the initial variables \mathbf{Z}_{in} via $\mathbf{Z}(t) = \mathcal{Q}(t)\mathbf{Z}_{in}$. Define the $2n \times 2n$ matrix \mathcal{L} as $\mathcal{L} = \lim_{t \rightarrow \infty} (\mathcal{Q}\tilde{\mathcal{Q}})^{1/2t}$, where $\tilde{\mathcal{Q}}$ is the matrix transpose of \mathcal{Q} . The LEs are then given by the logarithm of the eigenvalues of \mathcal{L} .

If we are going to view the oscillator variables in a statistical sense, however, then we are interested only in the Lyapunov spectrum, and in particular the maximal LE, of the system variable (x, p) given an ensemble of initial conditions for the oscillator variables (we will throughout this paper refer to the maximal LE simply as the LE). Note that because the system evolution is actively coupled to the environment variables, for each realization of a set of oscillator initial conditions, the system trajectory will itself be different. Moreover, as given explicitly in the definition of the noise term Eq. (4), any perturbations in the initial condition, $x(0)$, as required for defining the associated LE, would automatically change the realization of the noise force. This is an expected consequence of any systematic procedure as applied to a coupled Hamiltonian system. In our independent oscillator model, the noise is therefore a particularly simple case of *internal* noise.

III. SYSTEMS WITH EXTERNAL NOISE

We first consider the case of systems subjected to external noise. In this case, one assumes that the noise real-

izations are completely independent of the initial conditions of the system variables. This would be the straightforward interpretation of Eqs. (9) and (10) if we began our analysis with these two equations and the actual nature of their derivation was not specified. We will consider internal noise in the next section.

We consider an n -dimensional system in coordinate space, with the system trajectory written as $\mathbf{x}(t) \equiv (x_1(t), \dots, x_n(t))^T$. The (Gaussian, additive white noise) Langevin equation we consider here is essentially Eq. (9) written in a slightly generalized form,

$$\ddot{\mathbf{x}}(t) + \gamma \dot{\mathbf{x}}(t) = -\nabla U(\mathbf{x}(t)) + \sqrt{\mathcal{D}} dW_{ext}(t), \quad (12)$$

where $dW_{ext}(t)$ is the external noise term, with normalization set by $(dW_{ext})^2 = dt$, $U(\mathbf{x}(t))$ the potential, and \mathcal{D} an $n \times n$ diagonal matrix, with the k th diagonal entry, denoted D_k , being the noise intensity along x_k . Furthermore, γ is the damping coefficient ($\gamma > 0$), a time-independent scalar. Let $\mathbf{x}_0(t)$ be the fiducial trajectory, and write $\mathbf{x}(t) = \mathbf{x}_0(t) + \mathbf{\Delta}(t)$, where $\mathbf{\Delta}(t) \equiv (\Delta_1(t), \dots, \Delta_n(t))^T$. Upon linearization,

$$\ddot{\mathbf{\Delta}}(t) + \gamma \dot{\mathbf{\Delta}}(t) = -\mathcal{A}(t) \cdot \mathbf{\Delta}(t), \quad (13)$$

where $\mathcal{A}(t) \equiv (\partial_{x_i, x_j} U(\mathbf{x}_0(t)))$, the Jacobian of $\nabla U(\mathbf{x}_0(t))$. Notice that the noise term has disappeared in the linearized equation as we are considering it to be external. This means that any result we obtain will formally resemble that of the noise-free system; the only difference is that the noise may alter the fiducial trajectory $\mathbf{x}_0(t)$ and hence the time average of quantities that depend on it.

Now let

$$\mathbf{z}(t) = \mathbf{\Delta}(t)e^{\gamma t/2}, \quad (14)$$

and substitute into the linearized equation (13), to yield

$$\ddot{\mathbf{z}}(t) - \mathcal{B}(t) \cdot \mathbf{z}(t) = 0, \quad \text{where} \quad \mathcal{B}(t) = \frac{\gamma^2}{4} - \mathcal{A}(t). \quad (15)$$

Using a more canonical notation,

$$\mathbf{v}_1(t) = \mathbf{z}(t), \quad \mathbf{v}_2(t) = \dot{\mathbf{z}}(t), \quad (16)$$

Eq. 15 can be written as

$$\begin{pmatrix} \dot{\mathbf{v}}_1(t) \\ \dot{\mathbf{v}}_2(t) \end{pmatrix} = \mathcal{M}(t) \begin{pmatrix} \mathbf{v}_1(t) \\ \mathbf{v}_2(t) \end{pmatrix},$$

where $\mathcal{M}(t) \equiv \begin{pmatrix} \mathbf{0} & \mathcal{I} \\ \mathcal{B}(t) & \mathbf{0} \end{pmatrix}$. (17)

Notice here \mathcal{M} is a $2n \times 2n$ matrix. Since both $\mathbf{v}_1(t)$ and $\mathbf{v}_2(t)$ are n -dimensional column vectors, our above matrix equation is in fact a system of $2n$ equations. Its

formal solution is given by

$$\begin{aligned} \begin{pmatrix} \mathbf{v}_1(t) \\ \mathbf{v}_2(t) \end{pmatrix} &= \left(\mathcal{I} + \int_0^t \mathcal{M}(t_1) dt_1 \right. \\ &\quad + \int_0^t \int_0^{t_1} \mathcal{M}(t_1) \mathcal{M}(t_2) dt_2 dt_1 \\ &\quad \left. + \dots \right) \begin{pmatrix} \mathbf{v}_1(0) \\ \mathbf{v}_2(0) \end{pmatrix} \\ &\equiv \mathcal{Q}(t) \mathbf{v}(0), \end{aligned} \quad (18)$$

where $\mathcal{Q}(t) = Te^{\mathcal{M}(t)}$, the time-ordered exponential of $\mathcal{M}(t)$, and $\mathbf{v}(t) = (\mathbf{v}_1(t), \mathbf{v}_2(t))^T$.

Because long-time averages exist as stated in Section II, we can write $\mathcal{M}(t) = \bar{\mathcal{M}} + \mathcal{F}_M(t)$ where each entry of $\mathcal{F}_M(t)$ oscillates around zero. Denoting $\bar{\mathcal{Q}}(t) = Te^{\bar{\mathcal{M}}}$, our aim is now to show that, as far as computing the maximal LE is concerned, we can ignore the contribution $\mathcal{F}_M(t)$ and be able to substitute $\bar{\mathcal{Q}}(t)$ for $\mathcal{Q}(t)$ in Eq. (18). To justify such a substitution, we first use Eqs. (14) and (16) to write Eq. (18) as

$$\begin{aligned} \begin{pmatrix} \Delta(t) \\ \dot{\Delta}(t) + \frac{\gamma}{2} \Delta(t) \end{pmatrix} &= e^{-\gamma t/2} \mathcal{Q}(t) \mathbf{v}(0) \\ &= e^{-\gamma t/2} \bar{\mathcal{Q}}(t) \mathcal{R}(t) \mathbf{v}(0) \end{aligned} \quad (19)$$

where we define $\mathcal{R}(t)$ via $\bar{\mathcal{Q}}(t)^{-1} \mathcal{Q}(t) \equiv \mathcal{R}(t)$ (we assume $\bar{\mathcal{Q}}(t)$ is invertible). Every entry in $\mathcal{R}(t)$ grows slower than a linear exponential; to see this, note that

$$\begin{aligned} \mathcal{R}(t) &= \bar{\mathcal{Q}}(t)^{-1} \mathcal{Q}(t) \\ &= Te^{\mathcal{F}_M(t) + HOT(\bar{\mathcal{M}}, \mathcal{F}_M(t))} \end{aligned} \quad (20)$$

where the terms designated as *HOT* are given by the Campbell-Baker-Hausdorff series. Since $\bar{\mathcal{M}}$ is a bounded constant and $\mathcal{F}_M(t)$ is a bounded oscillating function, if we consider the RHS of the above equation as a matrix, then every entry in the matrix grows slower than a linear exponential.

Without loss of generality, let $|\Delta_{i_0}(t)|$ be the component of $\Delta(t)$ with the largest exponential dependence, and let $\bar{Q}_{i_0 j_0} R_{j_0 k_0}$ be the entry in $\bar{\mathcal{Q}}(t) \mathcal{R}(t)$ with the largest exponential dependence. Then the maximal LE of the system is given by

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{1}{t} \log \Delta_{i_0}(t) &= \lim_{t \rightarrow \infty} \frac{1}{t} \log \left(\sum_{j,k} \bar{Q}_{i_0 j} R_{j k} v_k(0) e^{-\gamma t/2} \right) \\ &= \lim_{t \rightarrow \infty} \frac{1}{t} \log (\bar{Q}_{i_0 j_0} R_{j_0 k_0}(t) v_{k_0}(0)) - \frac{\gamma}{2} \\ &= \lim_{t \rightarrow \infty} \frac{1}{t} \log (\bar{Q}_{i_0 j_0} v_{k_0}(0)) \\ &\quad + \lim_{t \rightarrow \infty} \frac{1}{t} \log R_{j_0 k_0}(t) - \frac{\gamma}{2} \\ &= \lim_{t \rightarrow \infty} \frac{1}{t} \log (\bar{Q}_{i_0 j_0} v_{k_0}(0)) - \frac{\gamma}{2} \\ &= \lim_{t \rightarrow \infty} \frac{1}{t} \log (\bar{Q}_{i_0 j_0} v_{j_0}(0)) - \frac{\gamma}{2} \end{aligned} \quad (21)$$

where in the second-to-last line we used the fact $R_{ij}(t)$ grows slower than a linear exponential, and hence $\lim_{t \rightarrow \infty} (1/t) \log R_{i_0 j_0}(t) = \lim_{t \rightarrow \infty} 1/t^{\epsilon(t)} = 0$, where $\epsilon(t) > 0$. In the last line we used the fact $v_{k_0}(0)$ is just a constant term and hence we can simply replace it with the constant term $v_{j_0}(0)$ without affecting the exponential behavior. Furthermore, since $\bar{\mathcal{Q}}(t) = Te^{\bar{\mathcal{M}}} = e^{\bar{\mathcal{M}}t}$, $\bar{\mathcal{Q}}(t)$ is a linear exponential of t . Thus, as $\bar{Q}_{i_0 j_0} R_{j_0 k_0}$ is the term in $\bar{\mathcal{Q}}(t) \mathcal{R}(t)$ with the largest exponential dependence, and all the entries of $\mathcal{R}(t)$ grow slower than a linear exponential, $\bar{Q}_{i_0 j_0}$ must be the term in $\bar{\mathcal{Q}}(t)$ with the largest exponential dependence. Therefore, the final line in the expression above is just the LE calculated had we replaced $\mathcal{Q}(t)$ with $\bar{\mathcal{Q}}(t)$.

Using the above arguments, we can substitute $\mathcal{Q}(t)$ as $\bar{\mathcal{Q}}(t)$ without affecting the LE. Hence we may write Eq. (18) as

$$\begin{pmatrix} \mathbf{v}_1(t) \\ \mathbf{v}_2(t) \end{pmatrix} \sim Te^{\bar{\mathcal{M}}} \begin{pmatrix} \mathbf{v}_1(0) \\ \mathbf{v}_2(0) \end{pmatrix}. \quad (22)$$

Furthermore, since $Te^{\bar{\mathcal{M}}} = e^{\bar{\mathcal{M}}t}$ as $\bar{\mathcal{M}}$ is constant, we have

$$\begin{pmatrix} \mathbf{v}_1(t) \\ \mathbf{v}_2(t) \end{pmatrix} \sim \exp \left[\begin{pmatrix} \mathbf{0} & \mathcal{I} \\ \bar{\mathcal{B}} & \mathbf{0} \end{pmatrix} t \right] \begin{pmatrix} \mathbf{v}_1(0) \\ \mathbf{v}_2(0) \end{pmatrix}. \quad (23)$$

Direct expansion of the matrix exponential yields

$$\exp \left[\begin{pmatrix} \mathbf{0} & \mathcal{I} \\ \bar{\mathcal{B}} & \mathbf{0} \end{pmatrix} t \right] = \begin{pmatrix} \cosh(\sqrt{\bar{\mathcal{B}}}t) & \sqrt{\bar{\mathcal{B}}}^{-1} \sinh(\sqrt{\bar{\mathcal{B}}}t) \\ \sqrt{\bar{\mathcal{B}}} \sinh(\sqrt{\bar{\mathcal{B}}}t) & \cosh(\sqrt{\bar{\mathcal{B}}}t) \end{pmatrix}, \quad (24)$$

where $\sqrt{\bar{\mathcal{B}}}$ is the matrix square root of $\bar{\mathcal{B}}$. Substituting Eqs. (14) and (16) into Eq. (23) and using Eq. (24), we get

$$\begin{aligned} \Delta(t) &\sim \cosh(\sqrt{\bar{\mathcal{B}}}t) e^{-\gamma t/2} \mathbf{v}_1(0) \\ &\quad + \sqrt{\bar{\mathcal{B}}}^{-1} \sinh(\sqrt{\bar{\mathcal{B}}}t) e^{-\gamma t/2} \mathbf{v}_2(0). \end{aligned} \quad (25)$$

To calculate $\sqrt{\bar{\mathcal{B}}}$, note that \mathcal{A} , given in Eq. (13), is a real symmetrical matrix, so \mathcal{B} (and hence $\bar{\mathcal{B}}$), defined in Eq. (15), is also a real symmetrical matrix and is hence diagonalizable with real eigenvalues. Then we can write $\bar{\mathcal{B}} = \mathcal{V} \mathcal{D} \mathcal{V}^{-1}$, where \mathcal{D} is a diagonal matrix. It follows that $\sqrt{\bar{\mathcal{B}}} = \mathcal{V} \sqrt{\mathcal{D}} \mathcal{V}^{-1}$, and $\sqrt{\mathcal{D}}$ is just the square root of the entries along the diagonal of \mathcal{D} . Thus, we see from Eq. (25) that if we want $\Delta(t)$ to have exponential divergence, we require $e^{\pm \sqrt{\bar{\mathcal{B}}}t}$ to have an exponential with power greater than $\gamma/2$ as one of its terms. In particular,

$$e^{\pm \sqrt{\bar{\mathcal{B}}}t} = e^{\pm \mathcal{V} \sqrt{\mathcal{D}} t \mathcal{V}^{-1}} = \mathcal{V} e^{\pm \sqrt{\mathcal{D}} t} \mathcal{V}^{-1}, \quad (26)$$

so we would like one of the diagonal terms of $\sqrt{\mathcal{D}}$, i.e. an eigenvalue of $\sqrt{\bar{\mathcal{B}}}$, to be larger than $\gamma/2$ in magnitude. Hence, $\bar{\mathcal{B}}$ must have an eigenvalue greater than $\gamma^2/4$ for the LE to be positive.

By definition, $\bar{\mathcal{B}} = \gamma^2/4 - \bar{\mathcal{A}}$, so if λ_B is an eigenvalue of $\bar{\mathcal{B}}$, then $\lambda_A = \gamma^2/4 - \lambda_B$ is an eigenvalue of $\bar{\mathcal{A}}$. Since

we need $\lambda_B > \gamma^2/4$ for the LE to be positive, this means we need $\lambda_A < 0$. However, the system is in thermal equilibrium, so

$$\begin{aligned} \overline{\partial_{x_i x_i}^2 U(\mathbf{x}_0(t))} &= C \int_{\Sigma} \partial_{x_i x_i}^2 U(\mathbf{x}) e^{-U(\mathbf{x})/D_i} dV \\ &= C \int_S \partial_{x_i} U(\mathbf{x}) e^{-U(\mathbf{x})/D_i} d\sigma \\ &\quad + \frac{C}{D_i} \int_{\Sigma} \partial_{x_i} U(\mathbf{x})^2 e^{-U(\mathbf{x})/D_i} dV, \end{aligned} \quad (27)$$

where C is the normalization constant, S the (fluctuating) constant-temperature hypersurface, and Σ the region enclosed by S . Because we assumed $\bar{\mathcal{B}}$, and hence $\bar{\mathcal{A}}$, is constant, the LHS of Eq. (27) is by definition just A_{ii} . Moreover, x_i is bounded by S , so increasing x_i for any i will lead to higher potential energy, which means $\partial_{x_i} U(\bar{x})$ is nonnegative on S . It follows the RHS of Eq. (27) is nonnegative, so $A_{ii} \geq 0$. Consequently, all the diagonal entries of A must be positive, and in particular $\text{tr } \bar{\mathcal{A}} \geq 0$.

It is now clear why the LE is always nonpositive in the 1-dimensional case, even with an external noise drive. In the 1-dimensional case, $\bar{\mathcal{A}}$ is a scalar, so it is trivially true that $\bar{\mathcal{A}} = \text{tr } \bar{\mathcal{A}} = \lambda_A$. However, as stated above, in order for the LE to be positive, we need $\lambda_A < 0$, while from Eq. (27), $\text{tr } \bar{\mathcal{A}} \geq 0$. Hence, there are simply not enough degrees of freedom in one dimension to satisfy these two conditions simultaneously and produce chaos. Of course, this result is not entirely surprising since it is well known that 1-dimensional Hamiltonian systems are not chaotic (although driven ones can be). Nonetheless, the calculation shows that in 1-dimensional systems with noise-induced chaos, either the noise source is not external, or the noise is such that the assumptions we made no longer hold, e.g., lack of a fluctuation-dissipation relation.

On the other hand, let us consider a 2-dimensional system. Then the eigenvalues of $\bar{\mathcal{A}}$ are

$$\lambda_A = \frac{\text{tr } \bar{\mathcal{A}} \pm \sqrt{(\text{tr } \bar{\mathcal{A}})^2 - 4 \det \bar{\mathcal{A}}}}{2}. \quad (28)$$

In order to have a positive LE, one of the solutions must be negative. This means it is necessary for either $\text{tr } \bar{\mathcal{A}} < 0$ or $\det \bar{\mathcal{A}} < 0$. We cannot have $\text{tr } \bar{\mathcal{A}} < 0$ by Eq. (27), but we can choose $\det \bar{\mathcal{A}} < 0$. A simple numerical illustration of this is

$$\bar{\mathcal{A}} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}. \quad (29)$$

With the choice $\gamma = 2$, indeed one of the exponents in Eq. (25) is positive. We would like to emphasize though that the entries in the above matrix $\bar{\mathcal{A}}$ are not the actual second derivatives of the potential; rather, they are the *time-averaged* values. If we naively treat the entries of $\bar{\mathcal{A}}$ as just the second derivatives of $U(\mathbf{x}_0(t))$, then the potential is the quadratic potential $U(x_1, x_2) = (1/2)x_1^2 + (1/2)x_2^2 + 2x_1x_2$, which is unbounded below

along the line $x_1 = -x_2$ and thus obviously chaotic (in an unbounded sense). However, if the entries in $\bar{\mathcal{A}}$ are the time-averaged values of $\partial_{x_i x_j} U(\mathbf{x}_0(t))$ for a particular unknown bounded potential, then the LE is positive for a system with weak noise.

We conclude this section with an explicit expression for the LE. We showed that the solution to the linearized version of the Langevin equation is given by Eq. (25):

$$\begin{aligned} \Delta(t) &\sim \cosh(\sqrt{\bar{\mathcal{B}}}t) e^{-\gamma t/2} \mathbf{v}_1(0) \\ &\quad + \sqrt{\bar{\mathcal{B}}}^{-1} \sinh(\sqrt{\bar{\mathcal{B}}}t) e^{-\gamma t/2} \mathbf{v}_2(0). \end{aligned} \quad (30)$$

where $\mathbf{v}_1(0) = \Delta(0)$ and $\mathbf{v}_2(0) = \dot{\Delta}(0) + (\gamma/2)\Delta(0)$, and $\bar{\mathcal{B}}$ is given by Eq. (15). Note every eigenvalue of $\bar{\mathcal{B}}$ is real as it's a real symmetric matrix, so every eigenvalue of $\sqrt{\bar{\mathcal{B}}}$ is purely real or purely imaginary. Let $\lambda_{\sqrt{\bar{\mathcal{B}}+}}$ be the maximum *real* eigenvalue of $\sqrt{\bar{\mathcal{B}}}$, and $\lambda_{\sqrt{\bar{\mathcal{B}}-}}$ be the minimum *real* eigenvalue of $\sqrt{\bar{\mathcal{B}}}$, and let $\lambda_{max} = \max(|\lambda_{\sqrt{\bar{\mathcal{B}}+}|, |\lambda_{\sqrt{\bar{\mathcal{B}}-}}|)$. Then we have the following cases:

1. If $\lambda_{max} > \gamma/2$, then $\Delta(t) \sim e^{(\lambda_{max} - \gamma/2)t}$ for large t , so the LE is positive.
2. If $\lambda_{max} = \gamma/2$, then $\Delta(t) \sim K$ for some constant K , so the LE is 0.
3. If $\lambda_{max} < \gamma/2$ or all the eigenvalues are imaginary, then $\Delta(t) \sim e^{(\lambda_{max} - \gamma/2)t}$ for large t , so the LE is negative.

As we can see from above, the damping coefficient is “suppressing” chaos, since if we take $\gamma \rightarrow 0$, then a system with a very small positive λ_{max} is chaotic. But if we slightly increase γ and presume the long time average $\bar{\mathcal{A}}(t)$, and hence λ_{max} is not affected, then the LE is smaller in this new system. We remark that for external noises, the noise term is decoupled from the damping mechanism. Therefore, the damping mechanism is associated just with the system, so the system is Hamiltonian only if the damping coefficient γ is zero. In this case, the third possibility given above is impossible as $\lambda_{max} \geq 0$ by construction. As our Hamiltonian system remains Hamiltonian after coupling it to a Hamiltonian heat bath, this shows that our conclusion is consistent with the fact that for Hamiltonian systems, there are equal number of positive and negative LEs in the spectrum, so the maximal LE is always nonnegative, i.e. only the first and second options above are valid.

IV. SYSTEMS WITH INTERNAL NOISE

We now proceed to examine internal noise. As before, the Langevin equation is

$$\ddot{\mathbf{x}}(t) + \gamma \dot{\mathbf{x}}(t) = -\nabla U(\mathbf{x}(t)) + \sqrt{\mathcal{D}} dW_{int}(t), \quad (31)$$

where the only difference from Eq. (12) is the fact that $dW_{int}(t)$ is an additive internal noise term. (In principle, the coupling can be more complicated, but we ignore that here.) The noise term arises from coupling the system self-consistently to an external dynamic structure. As discussed in the Introduction, these degrees of freedom could be a dynamical model for a heat bath, such as a collection of harmonic oscillators, or more generally, ‘fast’ modes in a system coupled to slower modes of physical interest. In any case, let us suppose there are m such outside structures. Then we can label the position and momentum variables of these outside structures as $\mathbf{q}(t) = (\mathbf{q}_1(t), \dots, \mathbf{q}_m(t))^T$ and $\mathbf{p}(t) = (\mathbf{p}_1(t), \dots, \mathbf{p}_m(t))^T$, respectively. Note that each \mathbf{q}_i and \mathbf{p}_j is an n -dimensional vector representing the n -dimensional space the system is in. As always, we remember that \mathbf{x} is an n -dimensional column vector, γ is a scalar, and \mathcal{D} is an $n \times n$ diagonal matrix, with the k th diagonal entry, denoted D_k , being the noise intensity along x_k .

We now emphasize that since we have internal noise, the term $dW_{int}(t)$ is inherently dependent on both the system initial conditions, $\mathbf{x}(0)$ and $\dot{\mathbf{x}}(0)$, and the ‘noise’ initial conditions, $\mathbf{q}(0)$ and $\mathbf{p}(0)$. This time, when we linearize the equation, we must decide to either only perturb the system initial conditions or those for both the system and noise. But perturbing the system initial conditions is the same as perturbing both the system and the noise as they are coupled, therefore, it doesn’t matter which one we choose. This does not mean that coupling a system to a noise source cannot change the system’s Lyapunov exponent. Rather, we are stating that for a system already coupled to an internal noise source, the exponent obtained by perturbing the system initial conditions is the same as that obtained by perturbing the initial conditions of both the system and the noise. For the rest of the section we will perturb initial conditions for both the system and noise during the linearization process.

Next, we note that here ‘noise’ is a term we use for a complicated underlying process with unknown exact behavior. Hence, for a particular internal Gaussian white noise realization $dW_{int}(t)$, we can write

$$\sqrt{\mathcal{D}} dW_{int}(t) = \sum_i \mathbf{h}_i(\mathbf{p}(0), \mathbf{q}(0), \mathbf{x}(0), \dot{\mathbf{x}}(0), t), \quad (32)$$

where \mathbf{h}_i ’s are unknown n -dimensional column vectors involving the initial conditions, such that when averaged over noise realizations (denoted as $\langle \dots \rangle_n$),

$$\begin{aligned} \langle \sqrt{D_i} dW_{int}(t) \rangle_n &= 0, \\ \langle \sqrt{D_i} dW_{int}(t) \sqrt{D_j} dW_{int}(t') \rangle_n &= \sqrt{D_i D_j} \delta(t - t') \end{aligned} \quad (33)$$

for all i, j . Now when we linearize Eq. (31), we perturb the initial values $\mathbf{x}(0)$, $\dot{\mathbf{x}}(0)$, $\mathbf{q}(0)$, and $\mathbf{p}(0)$. Denote the perturbed variables $\tilde{\Delta}(t)$, $\tilde{\dot{\Delta}}(t)$, $\tilde{\Delta}_q(t)$, and $\tilde{\Delta}_p(t)$, respectively, and let $\mathbf{x}_0(t)$ be the fiducial trajectory. Hence, the linearized equation is

$$\ddot{\tilde{\Delta}}(t) + \dot{\tilde{\Delta}}(t) = -\mathcal{A}(t)\tilde{\Delta}(t) + \delta(\sqrt{\mathcal{D}} dW_{int}(t)), \quad (34)$$

where $\mathcal{A}(t) \equiv (\partial_{x_i x_j} U(\mathbf{x}_0(t)))$, the Jacobian of $\nabla U(\mathbf{x}_0(t))$, and

$$\delta(\sqrt{\mathcal{D}} dW_{int}(t)) = \sum_i \Pi_i(\tilde{\Delta}) \quad (35)$$

is the perturbation of the noise term. Here, $\tilde{\Delta} = (I, t)$, where I denotes the initial conditions for the system and the noise such as $\tilde{\Delta}_p(0)$ and $\tilde{\mathbf{p}}(0)$. Note that every term in Π_i must, to first order (i.e. we Taylor expand functions to first order), be proportional to one of the perturbed initial conditions, as all the terms not proportional to an initial condition have canceled out from the linearization. Furthermore, note that once we linearized the equation, we no longer have a stochastic ODE. The reason is because our perturbations are exact, so even if in Eq. (31), we only knew the distribution of $\mathbf{q}(0)$ and $\mathbf{p}(0)$, in Eq. (34), we chose exact perturbations of the initial conditions. This means that $\delta(\sqrt{\mathcal{D}} dW_{int}(t))$ is not white noise!

Now, we don’t actually know the exact forms of the Π_i ’s, and without knowledge of these functions, we cannot solve the linearized ODE Eq. (34). Therefore, rather than letting the perturbations be exact, let us allow the perturbations to be a distribution such that $\delta(\sqrt{\mathcal{D}} dW_{int}(t))$ is Gaussian white noise when averaged over all possible perturbations. In other words, we choose the distribution to satisfy for all i, j, l, m

$$\langle \Pi_{il}(\tilde{\Delta}) \rangle_p = 0, \quad (36)$$

$$\sum_{i,j} \langle \Pi_{il}(\tilde{\Delta}) \Pi_{jm}(\tilde{\Delta}) \rangle_p = \sqrt{K_l K_m} \delta(t - t'), \quad (37)$$

where $\langle \dots \rangle_p$ denotes averaging over perturbations (so we can pull $\mathbf{x}(0)$, $\dot{\mathbf{x}}(0)$, $\mathbf{q}(0)$ and $\mathbf{p}(0)$ out as they don’t depend on noise perturbations). Here Π_{il} is the l th component of Π_i , and \mathcal{K} is the diagonal matrix analogous to \mathcal{D} . While the above two conditions can certainly be satisfied if the coupling is linear (so $\mathbf{h} \rightarrow \Pi$ by changing $\mathbf{x}(0)$, $\dot{\mathbf{x}}(0)$, $\mathbf{q}(0)$, $\mathbf{p}(0)$ to their perturbed quantities) by virtue of Eq. (33), it is unclear as to whether the above two conditions can be satisfied for all general couplings between system and noise.

We should point out that it is fine to take the perturbations along some directions to be zero, since perturbing any one initial condition causes the trajectory to be perturbed in all dimensions, as long as the variables are coupled. We will henceforth work under the assumption that Eqs. (36) and (37) can be satisfied in the system we’re working in. In particular, note that $\mathcal{K} \rightarrow 0$ as $\tilde{\Delta}(0)$, $\tilde{\dot{\Delta}}(0)$, $\tilde{\Delta}_q(0)$, $\tilde{\Delta}_p(0) \rightarrow 0$ by Eq. (37) and the fact that every term in Π_i is proportional to one of the perturbed initial conditions by the paragraph under Eq. (35). Thus, the distribution of the perturbations can be thought of as a white noise with infinitesimal intensity matrix \mathcal{K} . This means that when studying systems with internal noise, when we perturb the fiducial trajectory, the noise term in it is not simply a new noise realization

of intensity $\sqrt{\mathcal{D}}$, as done in Ref. [21]; rather, it has an infinitesimal intensity.

By viewing the perturbations as distributions, we can examine the mean and variance of the divergence behavior of perturbed trajectories due to slightly different perturbations. Note that for calculating the LE of a system, we don't care what the perturbation is, as long as it's infinitesimal and not specified only in one direction (the latter requirement makes sure we have a generic perturbation so the direction of the maximum exponent is perturbed); this is Osledec's Theorem. We emphasize that there is nothing special about viewing the initial perturbation as a distribution. We could have done the same in a noise-free system or a system with external noise. However, in both of these cases, to do so wouldn't change anything, as the linearized equation is Eq. (13)

$$\ddot{\vec{\Delta}}(t) + \gamma \dot{\vec{\Delta}}(t) = -\mathcal{A}(t)\vec{\Delta}(t), \quad (38)$$

where the initial conditions do not appear at all. It is true that the solution contains the initial conditions, so treating the initial perturbation as a distribution will now cause the solution to be a distribution; nonetheless, the exponential in the solution, and hence the LE, remains the same. Viewing the initial conditions as a distribution only helps if our system has internal noise, in which case we do not know all the terms in the linearized equation Eq. (34) and hence cannot evaluate it. Yet, because the initial perturbation is generic, letting it be a distribution does not change the LE of our system. Therefore, by examining the exponential behavior of the mean and standard deviation of each variable x_i , we can obtain the exact LE of the system with internal noise. The reason we cannot obtain the exponent through only the mean is because the mean divergence is effectively obtained by choosing a particular initial perturbation, which may not have been a generic perturbation and hence may not give the maximum exponent; the standard deviation, however, takes into account all generic perturbations.

From our arguments above, the linearized equation for our system with internal noise is

$$\ddot{\vec{\Delta}}(t) + \gamma \dot{\vec{\Delta}}(t) = -\mathcal{A}(t)\vec{\Delta}(t) + \sqrt{\mathcal{K}} dW(t), \quad (39)$$

where $\mathcal{A}(t) \equiv (\partial_{x_i x_j} U(x_0(t)))$ as before and $dW(t)$ is our perturbation distribution, which is Gaussian white noise. Again let

$$\mathbf{z}(t) = \vec{\Delta}(t)e^{\gamma t/2}, \quad (40)$$

so substitution yields

$$\begin{aligned} \ddot{\mathbf{z}}(t) - \mathcal{B}(t)\mathbf{z}(t) &= \sqrt{\mathcal{K}} dW(t)e^{\gamma t/2}, \\ \text{where } \mathcal{B}(t) &= \frac{\gamma^2}{4} - \mathcal{A}(t). \end{aligned} \quad (41)$$

As before, let

$$\mathbf{v}_1(t) = \mathbf{z}(t), \quad \mathbf{v}_2(t) = \dot{\mathbf{z}}(t), \quad (42)$$

and denote $\mathbf{v}(t) = (\mathbf{v}_1(t), \mathbf{v}_2(t))^T$. Then our second order ODE becomes

$$\begin{aligned} \frac{d}{dt} \begin{pmatrix} \mathbf{v}_1(t) \\ \mathbf{v}_2(t) \end{pmatrix} - \mathcal{M}(t) \begin{pmatrix} \mathbf{v}_1(t) \\ \mathbf{v}_2(t) \end{pmatrix} &= \begin{pmatrix} 0 \\ \sqrt{\mathcal{K}} dW(t)e^{\gamma t/2} \end{pmatrix} \\ &\equiv \mathbf{\Gamma}(t), \end{aligned} \quad (43)$$

where

$$\mathcal{M}(t) = \begin{pmatrix} \mathbf{0} & \mathcal{I} \\ \mathcal{B}(t) & \mathbf{0} \end{pmatrix}. \quad (44)$$

We keep in mind that $\mathcal{M}(t)$ is in fact a $2n \times 2n$ matrix, and $\mathbf{\Gamma}(t)$ is a $2n$ -dimensional column vector. As $\mathbf{\Gamma}(t)$ is Gaussian white noise, for all i, j ,

$$\langle \Gamma_i(t) \rangle_p = 0, \quad \langle \Gamma_i(t) \Gamma_j(t') \rangle_p = U_{ij}(t, t') \delta(t - t'), \quad (45)$$

where

$$\begin{aligned} \mathcal{U}(t, t') &= \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \Xi(t, t') \end{pmatrix}, \\ \Xi(t, t') &= \begin{pmatrix} K_1 & \sqrt{K_1 K_2} & \cdots & \sqrt{K_1 K_n} \\ \sqrt{K_2 K_1} & K_2 & \cdots & \sqrt{K_2 K_n} \\ \vdots & \vdots & \ddots & \vdots \\ \sqrt{K_n K_1} & \sqrt{K_n K_2} & \cdots & K_n \end{pmatrix} e^{\gamma(t+t')/2}. \end{aligned} \quad (46)$$

Here, \mathcal{U} is a $2n \times 2n$ matrix, with $U_{ij}(t, t') = U_{ji}(t, t')$, so all the conditions for a multivariate Gaussian white noise are satisfied [35].

We now invoke the assumption that the trajectories are sampling an equilibrium distribution over long times, so that $\mathcal{B}(t) \equiv \mathcal{B}$ and thus $\mathcal{M}(t) \equiv \mathcal{M}$ are constants. To solve Eq. (43), we first determine the homogeneous solution $\mathbf{v}^h(t)$:

$$\begin{aligned} \frac{d}{dt} \mathbf{v}^h(t) - \mathcal{M} \mathbf{v}^h(t) &= \mathbf{0} \\ \Rightarrow \mathbf{v}^h(t) &= e^{\mathcal{M}t} \mathbf{v}^h(0). \end{aligned} \quad (47)$$

We have already expanded this solution in the previous section. Next, we want a particular solution for the inhomogeneous case of Eq. (43). We will follow the steps given in Ref. [35] and write

$$\mathcal{G}(t) = e^{\mathcal{M}t}. \quad (48)$$

Suppose $v_i^{inh}(t) = \sum_j G_{ij}(t) c_j(t)$ for some c_j 's. Then

$$\dot{v}_i^{inh}(t) = \sum_j \left(\dot{G}_{ij}(t) c_j(t) + G_{ij}(t) \dot{c}_j(t) \right). \quad (49)$$

Furthermore, by Eq. (43)

$$\begin{aligned} \dot{v}_i^{inh}(t) &= \Gamma_i(t) + \sum_j M_{ij} v_j^{inh}(t) \\ &= \Gamma_i(t) + \sum_j M_{ij} G_{jk}(t) c_k(t) \\ &= \Gamma_i(t) + \sum_j \dot{G}_{ij}(t) c_j(t), \end{aligned} \quad (50)$$

where we used Eq. (48) to obtain the last equality. Equating Eqs. (49) and (50) yields

$$\sum_j G_{ij}(t) \dot{c}_j(t) = \Gamma_i(t). \quad (51)$$

In matrix representation,

$$\mathcal{G}(t) \dot{\mathbf{c}}(t) = \mathbf{\Gamma}(t). \quad (52)$$

Thus,

$$\begin{aligned} \int_0^t \mathcal{G}(t) \dot{\mathbf{c}}(t') dt' &= \int_0^t \mathcal{G}(t) \mathcal{G}^{-1}(t') \mathcal{G}(t') \dot{\mathbf{c}}(t') dt' \\ &= \int_0^t \mathcal{G}(t) \mathcal{G}^{-1}(t') \mathbf{\Gamma}(t') dt'. \end{aligned} \quad (53)$$

The LHS is just $\mathbf{v}^{inh}(t) = \mathcal{G}(t) \mathbf{c}(t)$. Recalling Eq. (48), the complete solution to Eq. (43) is

$$\begin{aligned} \mathbf{v}(t) &= \mathbf{v}^h(t) + \mathbf{v}^{inh}(t) \\ &= e^{\mathcal{M}t} \mathbf{v}(0) + \int_0^t e^{\mathcal{M}(t-t')} \mathbf{\Gamma}(t') dt'. \end{aligned} \quad (54)$$

Taking the mean of both sides and applying Eq. (45) yields

$$\langle \mathbf{v}(t) \rangle_p = e^{\mathcal{M}t} \langle \mathbf{v}(0) \rangle_p, \quad (55)$$

so the mean of $\mathbf{v}(t)$ behaves in exactly the same way as in the case for systems with external noise (c.f. Eq. (22)). Furthermore, we want to calculate the variance of Eq. (54). Multiplying the i th and j th coordinate of $\mathbf{v}(t)$ using Eq. (54), where $i, j \leq 2n$, taking the average (see Ref. [35]), and applying Eq. (45), we obtain the variance

$$\begin{aligned} \sigma_{ij}(\mathbf{v}(t)) &= G_{ij}(t) \sigma_{ij}(\mathbf{v}(0)) \\ &+ \sum_{k,s} \int_0^t \int_0^t G_{ik}(t-t'_1) G_{js}(t-t'_2) \\ &\times \langle \Gamma_k(t'_1) \Gamma_s(t'_2) \rangle_p dt'_1 dt'_2 \\ &= G_{ij}(t) \sigma_{ij}(\mathbf{v}(0)) \\ &+ \sum_{k,s} \int_0^t \int_0^t G_{ik}(t-t'_1) G_{js}(t-t'_2) \\ &\times U_{ks}(t'_1, t'_2) \delta(t'_1 - t'_2) dt'_1 dt'_2 \\ &= G_{ij}(t) \sigma_{ij}(\mathbf{v}(0)) \\ &+ \sum_{k,s} \int_0^t G_{ik}(t-t') G_{js}(t-t') U_{ks}(t', t') dt'. \end{aligned} \quad (56)$$

Hence, the variance of the i th variable is

$$\begin{aligned} \sigma_{ii}(\mathbf{v}(t)) &= G_{ii}(t) \sigma_{ii}(\mathbf{v}(0)) \\ &+ \sum_{k,s} \int_0^t G_{ik}(t-t') G_{is}(t-t') U_{ks}(t', t') dt'. \end{aligned} \quad (57)$$

Now, by definition Eq. (48)

$$\begin{aligned} \mathcal{G}(t-t') &= e^{\mathcal{M}(t-t')} = \exp \left[\begin{pmatrix} \mathbf{0} & \mathcal{I} \\ \mathcal{B} & \mathbf{0} \end{pmatrix} (t-t') \right] \\ &= \begin{pmatrix} \cosh \sqrt{\mathcal{B}}(t-t') & \sqrt{\mathcal{B}}^{-1} \sinh \sqrt{\mathcal{B}}(t-t') \\ \sqrt{\mathcal{B}} \sinh \sqrt{\mathcal{B}}(t-t') & \cosh \sqrt{\mathcal{B}}(t-t') \end{pmatrix}. \end{aligned} \quad (58)$$

This is in fact a $2n \times 2n$ matrix, so G_{11} for example *does not* refer to $\cosh \sqrt{\mathcal{B}}(t-t')$, but to the upper-left entry of $\cosh \sqrt{\mathcal{B}}(t-t')$. From Eqs. (40) and (42), we have $\mathbf{\Delta}(t) = e^{-\gamma t/2} \mathbf{v}_1(t)$. Let us denote v_{1i} the i th coordinate of $\mathbf{v}_1(t)$. Since this is also the i th coordinate of $\mathbf{v}(t)$ as $\mathbf{v}(t) = (\mathbf{v}_1(t), \mathbf{v}_2(t))^T$, it follows that for $i \leq n$,

$$\begin{aligned} \sigma_{ii}(\mathbf{\Delta}(t)) &= \langle (\Delta_i(t) - \overline{\Delta_i(t)})^2 \rangle_p \\ &= \langle e^{-\gamma t} (v_{1i}(t) - \overline{v_{1i}(t)})^2 \rangle_p \\ &= \langle e^{-\gamma t} (v_i(t) - \overline{v_i(t)})^2 \rangle_p \\ &= e^{-\gamma t} \sigma_{ii}(v(t)). \end{aligned} \quad (59)$$

It follows for $i \leq n$, the variance of the i th component of $\mathbf{\Delta}(t)$ is

$$\begin{aligned} \sigma_{ii}(\mathbf{\Delta}(t)) &= G_{ii} \sigma_{ii}(\mathbf{v}(0)) e^{-\gamma t} \\ &+ e^{-\gamma t} \sum_{k,s} \int_0^t G_{ik}(t-t') G_{is}(t-t') U_{ks}(t', t') dt', \end{aligned} \quad (60)$$

where $\mathcal{G}(t-t')$ is determined by Eq. (58) and $\mathcal{U}(t', t')$ is determined by Eq. (46). The standard deviation of the i th component of $\mathbf{\Delta}$ is thus

$$\begin{aligned} \text{sd}_{ii}(\mathbf{\Delta}(t)) &= [G_{ii} \sigma_{ii}(\mathbf{v}(0)) e^{-\gamma t} \\ &+ \sum_{k,s} e^{-\gamma t} \int_0^t G_{ik}(t-t') G_{is}(t-t') U_{ks}(t', t') dt']^{1/2}. \end{aligned} \quad (61)$$

Let us use our previous notation, where $\lambda_{\sqrt{\mathcal{B}}+}$ and $\lambda_{\sqrt{\mathcal{B}}-}$ are the maximum and minimum real eigenvalues of $\sqrt{\mathcal{B}}$, respectively, and $\lambda_{max} = \max(|\lambda_{\sqrt{\mathcal{B}}+}|, |\lambda_{\sqrt{\mathcal{B}}-}|)$. Recall from Eq. (26) that

$$e^{\pm \sqrt{\mathcal{B}}t} = e^{\pm \mathcal{V} \sqrt{\mathcal{D}} t \mathcal{V}^{-1}} = \mathcal{V} e^{\pm \sqrt{\mathcal{D}} t} \mathcal{V}^{-1}. \quad (62)$$

Suppose λ_{max} is the i th entry of the diagonal matrix $\sqrt{\mathcal{D}}$. Then the i th row of $e^{\pm \sqrt{\mathcal{D}} t} \mathcal{V}^{-1}$ are all entries involving $e^{\lambda_{max}}$, where we choose the sign of $\sqrt{\mathcal{D}}$ so λ_{max} is nonnegative. Thus, every entry in $e^{\pm \sqrt{\mathcal{B}} t} = \mathcal{V} e^{\pm \sqrt{\mathcal{D}} t} \mathcal{V}^{-1}$ involves $e^{\lambda_{max}}$. Cancellation may occur for some terms if $\sqrt{\mathcal{B}}$ has multiple eigenvalues λ_{max} , but we do not expect cancellation to occur for all terms. Then for large t ,

$$G_{ij} \sim e^{\lambda_{max} t}, \quad (63)$$

for some i, j . Hence, at least one term in the sum on the RHS of Eq. (61) is proportional to

$$\begin{aligned}
& e^{-\gamma t} \int_0^t e^{2\lambda_{max}(t-t')} e^{\gamma t'} dt' \\
&= e^{(2\lambda_{max}-\gamma)t} \int_0^t e^{(\gamma-2\lambda_{max})t'} dt' \\
&= e^{(2\lambda_{max}-\gamma)t} \cdot \frac{1}{\gamma-2\lambda_{max}} \left(e^{(\gamma-2\lambda_{max})t} - 1 \right) \\
&\sim |e^{(2\lambda_{max}-\gamma)t} - 1|,
\end{aligned} \tag{64}$$

where we used Eq. (46) to conclude $U_{ij} \sim e^{\gamma t'}$. Hence, this term has a larger exponential than the first term on the RHS of Eq. (61) and dominates the expression. It follows that

$$\text{sd}_{ii}(\Delta(t)) \sim \sqrt{|e^{(2\lambda_{max}-\gamma)t} - 1|}. \tag{65}$$

Since the mean of $\Delta(t)$ is the RHS of Eq. (25) with different initial conditions, when $\lambda_{max} \geq \gamma/2$, the mean and standard deviation of Δ have the same exponential behavior, while if $\lambda_{max} < \gamma/2$ or λ_{max} does not exist (the eigenvalues are imaginary), the constant term in the standard deviation dominates. As in the external noise case, every eigenvalue of \sqrt{B} is either purely real or purely imaginary, and we then have the following cases:

1. If $\lambda_{max} > \gamma/2$, then $\text{sd}_{ii}(\Delta(t)) \sim e^{(\lambda_{max}-\gamma/2)t}$ for large t , so the LE is positive.
2. If $\lambda_{max} = \gamma/2$, then $\text{sd}_{ii}(\Delta(t)) \sim K$ for some constant K , so the LE is 0.
3. If $\lambda_{max} < \gamma/2$, then $\text{sd}_{ii}(\Delta(t)) \sim K$ for large t for some constant K , so the LE is 0.
4. If the eigenvalue(s) of \sqrt{B} are all imaginary, then $\text{sd}_{ii}(\Delta(t))$ is sinusoidal, so the LE is 0.

This strongly resembles the cases in the previous section, when we examined systems with external noise. However, we do note that the internal noise now prevents exponential convergence from occurring. Intuitively, this makes perfect sense, since while the damping term γ causes the phase space trajectories to converge, the noise contributes energy into the system and hence counteracts the damping. Clearly, since we are assuming the noise is weak, it cannot have an impact on the chaotic behavior of systems with positive LEs. However, if the system by itself has a negative exponent, then the internal noise will begin to dominate the separation of nearby trajectories after a long time, as the noise terms stay constant while the system terms become “weaker” due to damping. We would like to point out, however, that even with internal noise, we cannot have 1-dimensional chaos. This conclusion can be easily seen by the same argument used for the external noise case.

V. CONCLUSION

Our primary purpose here was to understand the basic issues in defining the LE for a noisy system, in a situation where a controlled analysis is possible. To do this we first provided a context where ‘noise’ is more or less clearly defined, by exploiting the oscillator heat bath paradigm. Although this paradigm is by no means completely general, it serves as an illustrative example for what, in principle, needs to be worked out in more complex situations. By using this model, we can define the LE in an uncontroversial way, by first linearizing around the dynamical trajectory, and only later considering what terms need to be thought of as noise, and under what circumstances.

We distinguished in our work between external and internal noise to avoid dynamical inconsistencies (see, e.g., the discussion in Ref. [3]). By setting up the definition of noise following the oscillator heat bath approach, we first considered the case of external noise as an uncontrolled limit of the model. Even in this case, the second-order nature of the equations of motion and the existence of a fluctuation-dissipation theorem helps us to arrive at reasonable conclusions about the behavior of the LEs – no chaos for one-dimensional systems, but the possibility remaining open in higher dimensions.

Turning next to the case of internal noise perturbations, we noted that the noise forcing terms in this case cannot be set to zero after linearization, because of the self-consistency requirement. A residual piece remains, proportional to a set of (unknown) initial conditions for the bath. In the case of a linear system-bath coupling we can proceed by constructing a particular distribution of initial conditions such that when averaged over it, the perturbations arising from the initial conditions do have the properties of Gaussian white noise, characterized by an infinitesimal noise intensity matrix \mathcal{K} . So, when the fiducial trajectory is perturbed, it is not perturbed by the original noise strength \sqrt{D} , which can be much larger. The late-time limit of the standard deviation of the perturbed trajectory ensemble can be used to find the maximal LE, using once again the fact that the trajectories are exploring a canonical distribution.

Although our analysis helped shed some light on the relationship between chaos and weak noise, it is only a small step towards understanding this complex yet fascinating relationship. For instance, our treatment was restricted to the case of thermal equilibrium, and we did not consider systems driven by external time-dependent forces. In principle, explicit time-dependences can be introduced into the oscillator models [37], but the possible lack of a stable late-time distribution will likely restrict the statements that can be made on an analytical basis.

Furthermore, our analysis also raises some possible implications on how noise induced chaos may arise. Noise-induced chaos is chaotic behavior in a system that arises only when the system is coupled to noise. From our remarks in the last paragraph of Section IV, there cannot be noise induced chaos in 1-dimensional ergodic (equi-

lbrium) systems, so let us consider the case of higher-dimensions. From Eq. (15) and taking $\mathcal{B}(t) \equiv \mathcal{B}$ constant, we see that $B_{ii} = \gamma^2/4 - A_{ii}$. But we also know from Eq. (27) that

$$A_{ii} = C \int_S \partial_{x_i} U(x) e^{-U(\vec{x})/D_i} d\sigma + \frac{C}{D_i} \int_\Sigma \partial_{x_i} U(\vec{x})^2 e^{-U(\vec{x})/D_i} dV, \quad (66)$$

so B_{ii} , and hence \mathcal{B} , is dependent on noise intensity. Because the eigenvalues of $\sqrt{\mathcal{B}}$ determine the LE of the system, this means if \mathcal{B} changes, the LE will potentially also change. In particular, let us suppose our system has a λ_{max} slightly less than $\gamma/2$. Since our analysis of systems with external noise formally resembles that of noise-free systems as the noise term disappears via linearization (see Eq. (13) and the paragraph below), this means if λ_{max} is slightly less than $\gamma/2$, then our system has a negative LE. Now, let us couple this system to ei-

ther external or internal noise. If the noise intensity D_i is small enough such that the system coupled to the noise is still approximately ergodic, but large enough to shift the eigenspectrum of $\sqrt{\mathcal{B}}$ such that λ_{max} is now slightly greater than $\gamma/2$, then our system now has a positive LE, resulting in noise-induced chaos. However, how to couple the noise so that such a shift in the eigenspectrum occurs is still an open question, and one that is worth exploring.

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- [1] J.-P. Eckmann and D. Ruelle, *Rev. Mod. Phys.* **57**, 617 (1985).
 - [2] P. Gaspard and J.R. Dorfman, *Phys. Rev. E* **52**, 3525 (1995).
 - [3] N.G. van Kampen, *Stochastic Processes in Physics and Chemistry* (North-Holland, Amsterdam, 1987).
 - [4] R.W. Zwanzig, *Nonequilibrium Statistical Mechanics* (Oxford University Press, New York, 2001).
 - [5] K. Jacobs, *Stochastic Processes for Physicists: Understanding Noisy Systems* (Cambridge University Press, Cambridge, 2010).
 - [6] R.W. Zwanzig, in *Systems Far From Equilibrium*, Proc. Sitges Conference on Statistical Mechanics, L. Garrido (ed.) (Springer-Verlag, Berlin, 1980).
 - [7] J.P. Crutchfield, J.D. Farmer, and B.A. Huberman, *Phys. Rep.* **92**, 46 (1982).
 - [8] J.B. Gao, S.K. Hwang, and J.M. Liu, *Phys. Rev. Lett.* **82**, 1132 (1999).
 - [9] S.K. Hwang, J.B. Gao, and J.M. Liu, *Phys. Rev. E* **61**, 5162 (2000).
 - [10] T. Tél, Y.-C. Lai, and M. Gruiz, *Int. J. Bifurcation and Chaos*, **18**, 509 (2008); and references therein.
 - [11] S.P. Ellner and P. Turchin, *Oikos* **111**, 620 (2005).
 - [12] S. Habib, H.E. Kandrup, and M.E. Mahon, *Astrophys. J.* **480**, 155 (1997); and references therein.
 - [13] See, e.g., T. Bhattacharya, S. Habib, and K. Jacobs, *Phys. Rev. Lett.* **85**, 4852 (2000); and references therein.
 - [14] S. Habib, K. Jacobs, and K. Shizume, *Phys. Rev. Lett.* **96**, 010403 (2006).
 - [15] A. Vulpiani, in *Proceedings of the International Workshop, Fluctuations in Physics and Biology: Stochastic Resonance, Signal Processing and Related Phenomena*, *Il Nuovo Cimento* **17**, 653 (1995).
 - [16] J.B. Gao, C.C. Chen, S.K. Hwang, and J.M. Liu, *Int. J. Mod. Phys. B* **13**, 3283 (1999).
 - [17] R.J. Rubin, *J. Math. Phys.* **1**, 309 (1960).
 - [18] G. Ford, M. Kac, and P. Mazur, *J. Math. Phys.* **6**, 504 (1965).
 - [19] R.W. Zwanzig, *J. Stat. Phys.* **9**, 215 (1973).
 - [20] A.O. Caldeira and A.J. Leggett, *Physica A* **121**, 587 (1983); *Ann. Phys.* **149**, 374 (1983).
 - [21] C. Van den Broeck and G. Nicolis, *Phys. Rev. E* **48**, 4845 (1993).
 - [22] V. Loreto, G. Paladin, and A. Vulpiani, *Phys. Rev. E* **53**, 2087 (1995).
 - [23] L.D. Landau, *Phys. Z. Sowjetunion* **10**, 154 (1936).
 - [24] R. Zwanzig, *J. Chem. Phys.* **33**, 1338 (1960).
 - [25] H. Mori, *Prog. Theor. Phys.* **33**, 423 (1965); **34**, 399 (1965).
 - [26] H. Grabert, *Projection Operator Techniques in Nonequilibrium Statistical Mechanics* (Springer-Verlag, Berlin, 1982).
 - [27] R. Kupferman and A.M. Stuart, *Physica D* **199**, 279 (2004).
 - [28] G. Ariel and E. Vanden-Eijnden, *Nonlinearity* **22**, 145 (2009).
 - [29] K. Lindenberg and V. Seshadri, *Physica A* **109**, 483 (1981).
 - [30] J. Qiang and S. Habib, *Phys. Rev. E* **62**, 7430 (2000).
 - [31] R.W. Zwanzig, in *Statistical mechanics, new concepts, new problems, new applications*, S.A. Rice, K.F. Freed, and J.C. Light (eds.) (University of Chicago Press, Chicago, 1972).
 - [32] E. Pollak, H. Grabert, and P. Hänggi, *J. Chem. Phys.* **91**, 4073 (1989).
 - [33] G. Ariel and E. Vanden-Eijnden, *J. Stat. Phys.* **126**, 43 (2007).
 - [34] R. Kupferman, *J. Stat. Phys.* **114**, 291 (2004).
 - [35] H. Risken, *The Fokker-Planck Equation* (Springer-Verlag, Berlin, 1989).
 - [36] C.W. Gardiner, *Handbook of Stochastic Methods* (Springer-Verlag, Berlin, 2004).
 - [37] S. Habib and H.E. Kandrup, *Phys. Rev. D* **46**, 5303 (1992).